Part 1: FIR Filters – Window Method

Tutorial ISCAS 2007

July 24, 2007

- A simple method for the design of FIR filters is through the use of the Fourier series in conjunction with the application of a class of functions known as window functions.

- Arbitrary specifications can be achieved by using a method proposed by Kaiser.

Note: The material for this module is taken from Antoniou, *Digital Signal Processing: Signals, Systems, and Filters,* Chap. 9.)

- A fundamental property of digital filters in general is that they have a periodic frequency response with period equal to the sampling frequency ω_s, i.e.,

$$H(e^{j(\omega+k\omega_s)T}) = H(e^{j\omega T})$$

- Therefore, an arbitrary desired frequency response, $H(e^{j\omega T})$, can be represented by a Fourier series as

$$H(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} h(nT)e^{-j\omega nT} \tag{A}$$

where $\quad h(nT) = \dfrac{1}{\omega_s} \displaystyle\int_{-\omega_s/2}^{\omega_s/2} H(e^{j\omega T})e^{j\omega nT}\, d\omega$

are the Fourier series coefficients.

• • •

$$H(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} h(nT)e^{-j\omega nT} \qquad \text{(A)}$$

If we let $e^{j\omega T} = z$ in Eq. (A), we get

$$H(z) = \sum_{n=\infty}^{\infty} h(nT)z^{-n}$$

This is the transfer function of an FIR filter with impulse response $h(nT)$.

Since the Fourier series coefficients are defined over the range $-\infty < n < \infty$, two problems are associated with the Fourier series method:

- The FIR filter obtained is of infinite length.

Since the Fourier series coefficients are defined over the range $-\infty < n < \infty$, two problems are associated with the Fourier series method:

- The FIR filter obtained is of infinite length.

- The filter is noncausal because the impulse response is nonzero for negative time.

- A finite filter length can be achieved by truncating the impulse response such that

$$h(nT) = 0 \quad \text{for} \ |n| > M$$

where $M = (N - 1)/2$.

- A finite filter length can be achieved by truncating the impulse response such that

$$h(nT) = 0 \quad \text{for} \ |n| > M$$

where $M = (N - 1)/2$.

- On the other hand, a causal filter can be obtained by delaying the impulse response by a period MT seconds or by M sampling periods.

- Since delaying the impulse response by M sampling periods amounts to multiplying the transfer function by z^{-M}, the transfer function of the causal filter assumes the form

$$H'(z) = z^{-M} \sum_{n=-M}^{M} h(nT)z^{-n}$$

- Since delaying the impulse response by *M* sampling periods amounts to multiplying the transfer function by z^{-M}, the transfer function of the causal filter assumes the form

$$H'(z) = z^{-M} \sum_{n=-M}^{M} h(nT)z^{-n}$$

- The frequency response of the causal filter is obtained by letting $z = e^{j\omega T}$ in the transfer function, i.e.,

$$H'(e^{j\omega T}) = e^{-jM\omega T} \sum_{n=-M}^{M} h(nT)e^{-jn\omega T}$$

and since $|e^{-jM\omega T}| = 1$, delaying the impulse response by *M* sampling periods does not change the amplitude response.

Example

Design a lowpass filter with a desired frequency response

$$H(e^{j\omega T}) \approx \begin{cases} 1 & \text{for } |\omega| \le \omega_c \\ 0 & \text{for } \omega_c < |\omega| \le \omega_s/2 \end{cases}$$
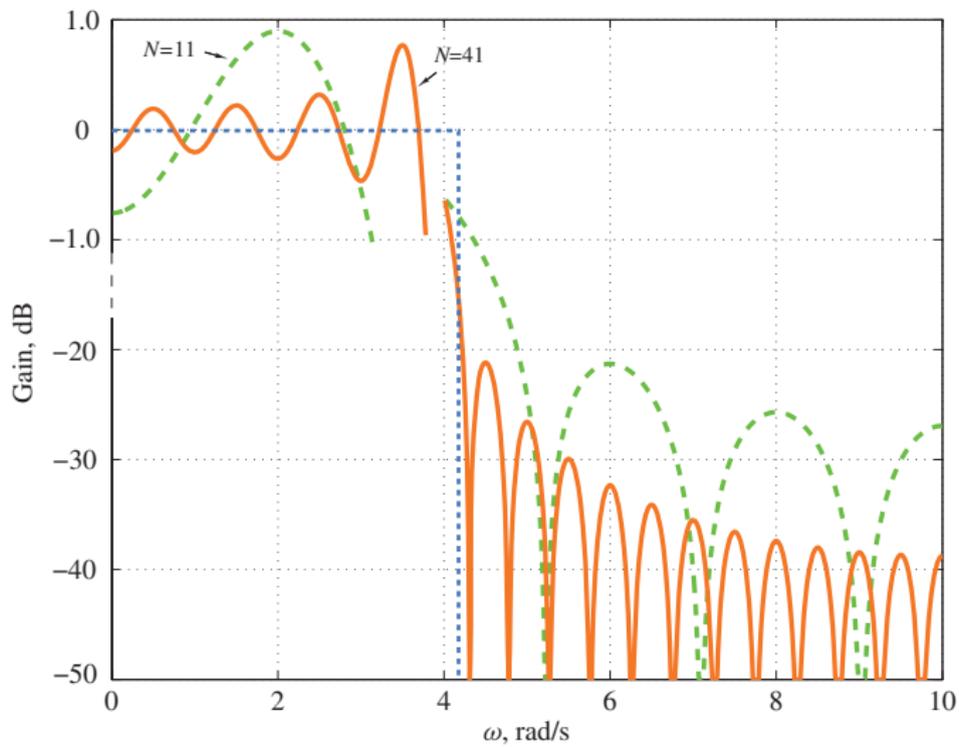
where ω_s is the sampling frequency.

Solution The Fourier series gives the impulse response of the noncausal filter as

$$h(nT) = \frac{1}{\omega_s} \int_{-\omega_c}^{\omega_c} e^{j\omega nT} \, d\omega = \frac{1}{\omega_s} \left[\frac{e^{j\omega nT}}{jnT} \right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{n\pi} \frac{(e^{j\omega_c nT} - e^{-j\omega_c nT})}{2j} = \frac{1}{n\pi} \sin \omega_c nT \qquad \text{(B)}$$

Truncating and delaying the impulse response by M sampling periods immediately yields the required design.

Example *Cont'd*

- The amplitude response of the filter (magnitude of the frequency response) exhibits oscillations in the passband as well as the stopband, which are known as Gibbs' oscillations. They are caused by the truncation of the Fourier series.

- The amplitude response of the filter (magnitude of the frequency response) exhibits oscillations in the passband as well as the stopband, which are known as Gibbs' oscillations. They are caused by the truncation of the Fourier series.

- As the filter length is increased, the frequency of the oscillations increases but the amplitude stays constant.

- The amplitude response of the filter (magnitude of the frequency response) exhibits oscillations in the passband as well as the stopband, which are known as Gibbs' oscillations. They are caused by the truncation of the Fourier series.

- As the filter length is increased, the frequency of the oscillations increases but the amplitude stays constant.

- In other words, we do not seem to be able to reduce the passband and stopband errors below a certain limit by increasing the filter length.

  Therefore, the filters that can be designed with the Fourier series method are of little practical usefulness.

- The standard technique for the reduction of Gibbs' oscillations is to truncate the infinite-duration impulse response, $h(nT)$, through the use of a discrete-time window function $w(nT)$.

- The standard technique for the reduction of Gibbs' oscillations is to truncate the infinite-duration impulse response, $h(nT)$, through the use of a discrete-time window function $w(nT)$.

- If we let

$$h_w(nT) = w(nT)h(nT)$$

then a modified transfer function is obtained as

$$\begin{aligned}
H_w(z) &= \mathcal{Z}[w(nT)h(nT)] \\
&= \frac{1}{2\pi j} \oint_\Gamma H(v) W\left(\frac{z}{v}\right) v^{-1}\, dv
\end{aligned}$$

where $H(z)$ is the original transfer function and $W(z)$ is the z transform of the window function.
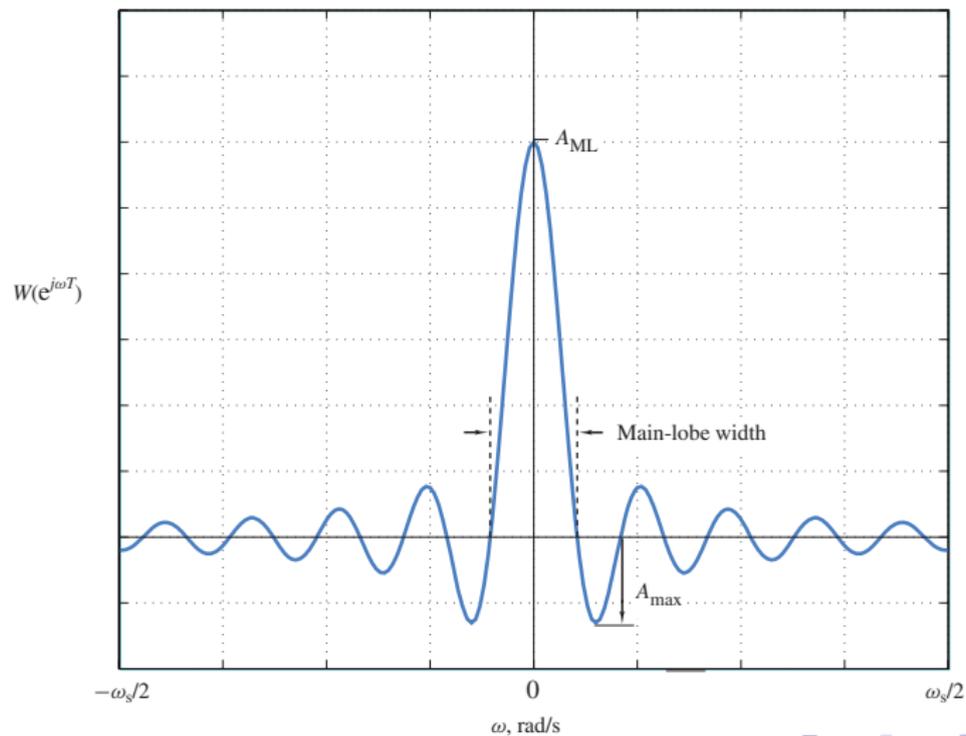
- Evaluating $H_w(z)$ on the unit circle $z = e^{j\omega T}$ gives the frequency response of the modified filter as

$$H_w(e^{j\omega T}) = \frac{T}{2\pi} \int_0^{2\pi/T} H(e^{j\varpi T}) W(e^{j(\omega-\varpi)T}) \, d\varpi \qquad \text{(C)}$$

  $W(e^{j\omega T})$ is the *frequency spectrum* of the window function and the integral at the right-hand side is a *convolution integral*.

Frequency spectrum of a typical window:

A. Antoniou   Part 1: FIR Filters – Window Method

- Windows are characterized by their *main-lobe width*, B_{ML}, which is the bandwidth between the first negative and the first positive zero crossings, and by their *ripple ratio*, r, which is defined as

$$r = 100\frac{A_{max}}{A_{ML}} \% \quad \text{or} \quad R = 20\log\frac{A_{max}}{A_{ML}} \text{ dB}$$

where A_{max} and A_{ML} are the maximum side-lobe and main-lobe amplitudes, respectively.

- Windows are characterized by their *main-lobe width*, B_{ML}, which is the bandwidth between the first negative and the first positive zero crossings, and by their *ripple ratio*, r, which is defined as

$$r = 100\frac{A_{max}}{A_{ML}} \% \quad \text{or} \quad R = 20\log\frac{A_{max}}{A_{ML}} \text{ dB}$$

where A_{max} and A_{ML} are the maximum side-lobe and main-lobe amplitudes, respectively.

- The main-lobe width and ripple ratio should be as low as possible, i.e., the spectral energy of the window should be concentrated as far as possible in the main lobe and the energy in the side lobes should be as low as possible.

- The way by which Gibbs' oscillations can be controlled by using a window is illustrated in the next few slides which are based on Eq. (C), i.e.,
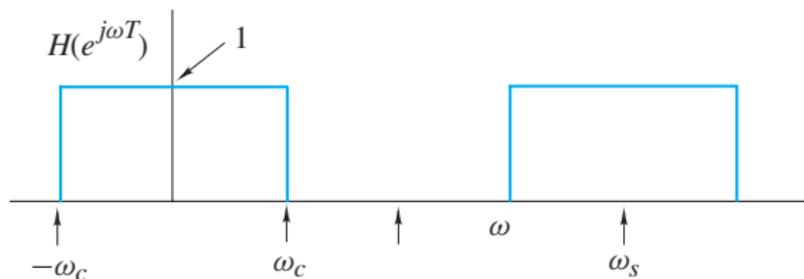
$$H_w(e^{j\omega T}) = \frac{T}{2\pi} \int_0^{2\pi/T} H(e^{j\varpi T}) W(e^{j(\omega-\varpi)T}) \, d\varpi \qquad \text{(C)}$$

- The way by which Gibbs' oscillations can be controlled by using a window is illustrated in the next few slides which are based on Eq. (C), i.e.,

$$H_w(e^{j\omega T}) = \frac{T}{2\pi} \int_0^{2\pi/T} H(e^{j\varpi T}) W(e^{j(\omega-\varpi)T}) \, d\varpi \qquad \text{(C)}$$

- Let us consider the application of a generic window in the design of a LP filter, assuming that the area under the curve in the frequency spectrum of the window is $2\pi/T$.
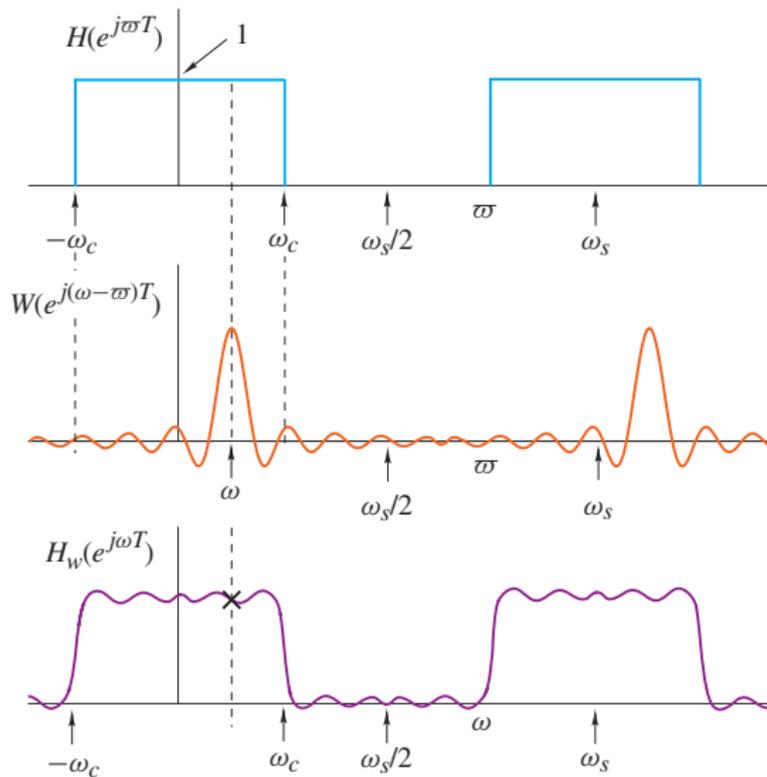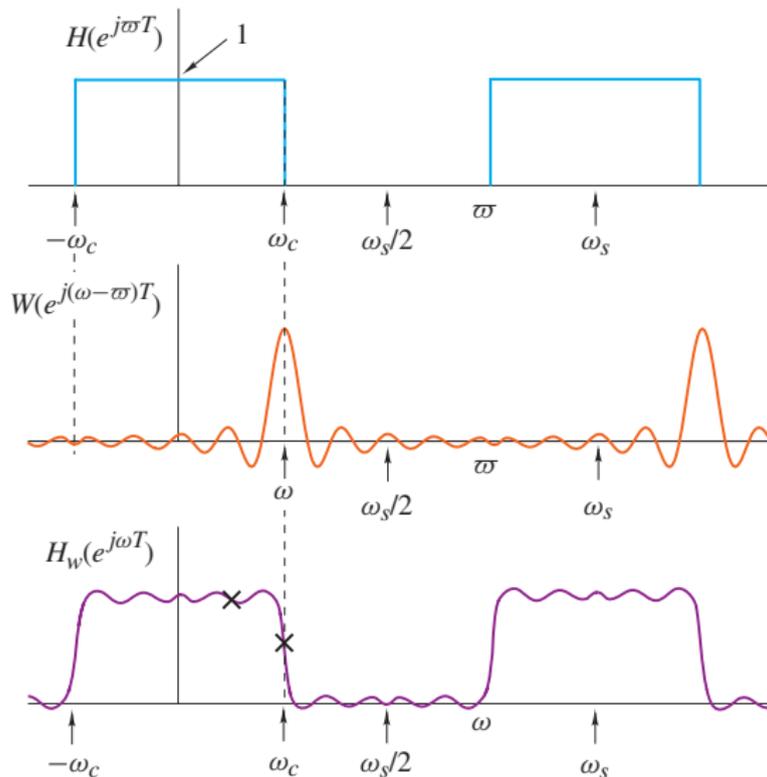
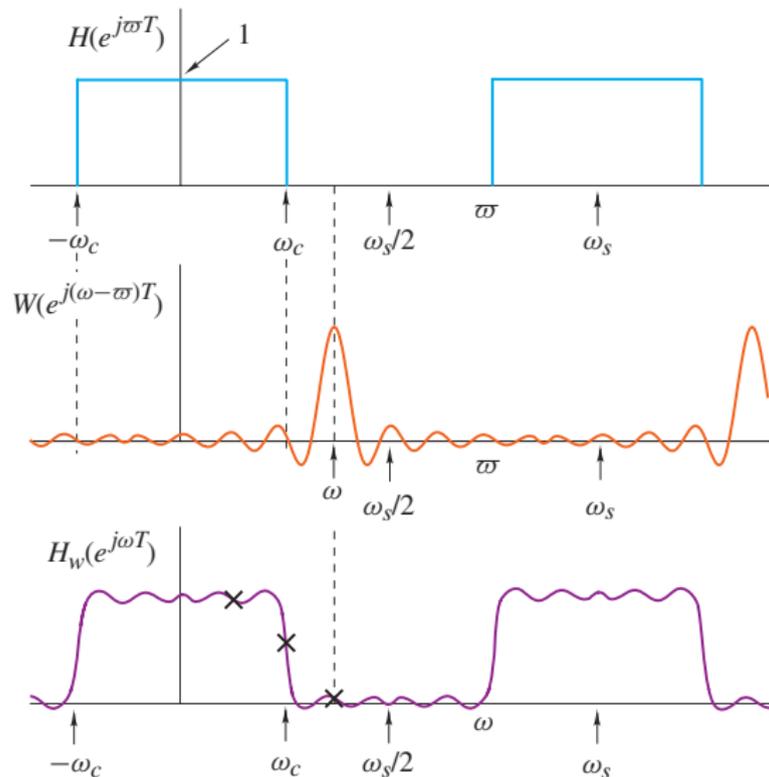Frequency response of ideal lowpass filter
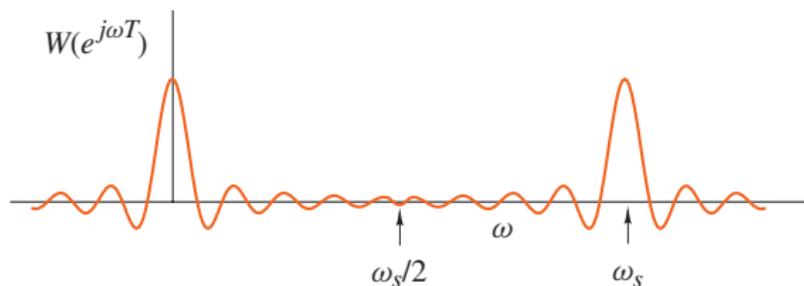
Frequency spectrum of window

From the illustrations, we conclude that

- the steepness of the transition characteristic of the filter obtained depends on the main-lobe width of the window, and

From the illustrations, we conclude that

- the steepness of the transition characteristic of the filter obtained depends on the main-lobe width of the window, and

- the amplitudes of the passband and stopband ripples depend on the ripple ratio of the window.

- One of the most important windows is the Kaiser window.

- One of the most important windows is the Kaiser window.

- This is a parametric window which has an independent control parameter α.

- One of the most important windows is the Kaiser window.

- This is a parametric window which has an independent control parameter α.

- By choosing the value of α and the filter length, N, arbitrary specifications can be achieved in lowpass (LP), highpass (HP), bandpass (BP), and bandstop (BS) filters.

Ripple ratio versus α:
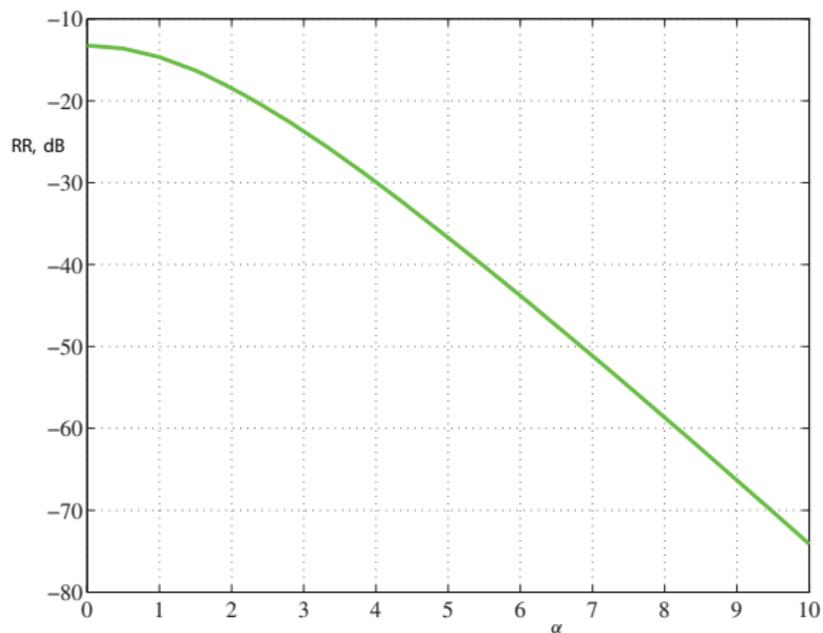
Main-lobe width versus α:
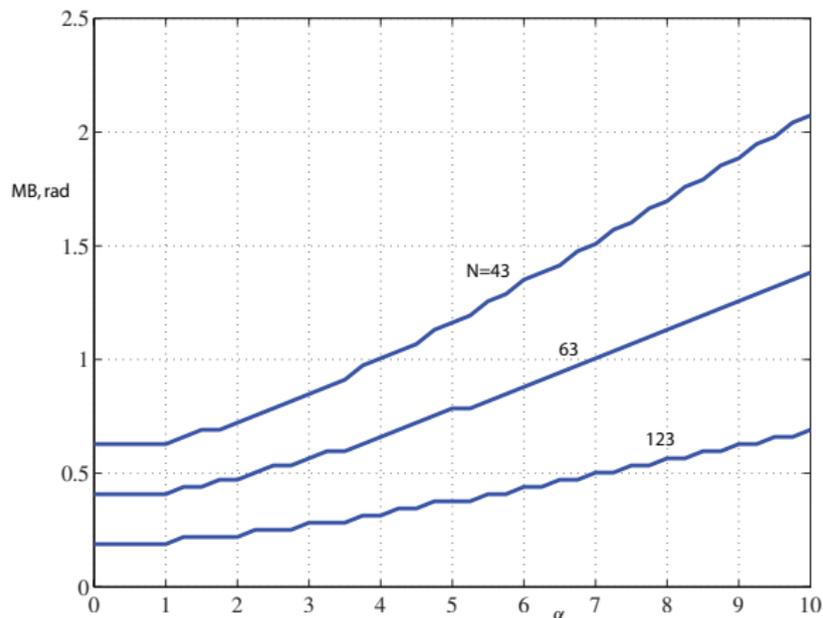
The Kaiser window function is given by

$$w_K(nT) = \begin{cases} \dfrac{I_0(\beta)}{I_0(\alpha)} & \text{for } |n| \leq M \\ 0 & \text{otherwise} \end{cases} \tag{D}$$

where
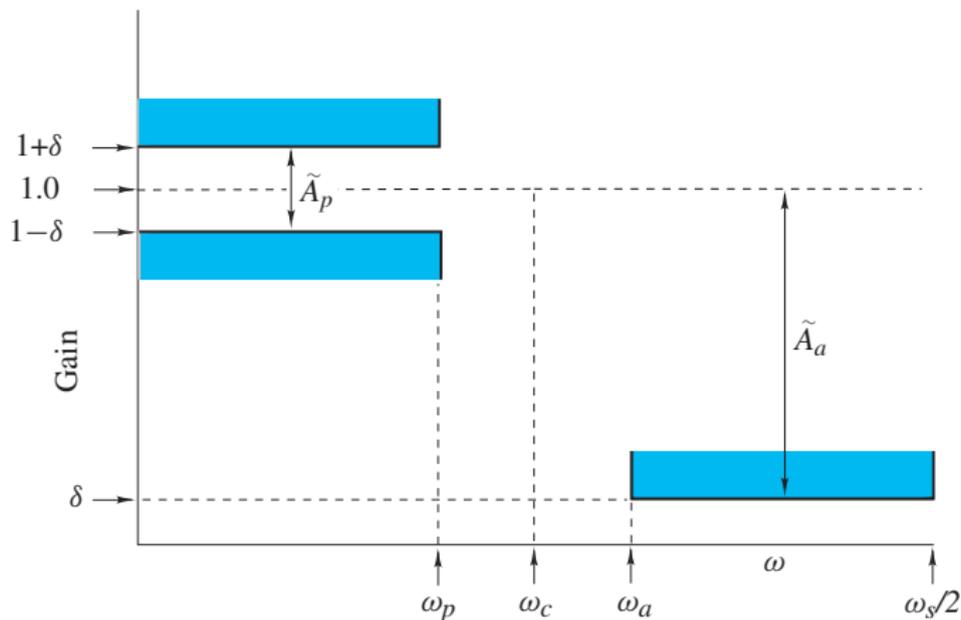
$$\beta = \alpha\sqrt{1 - \left(\frac{n}{M}\right)^2}, \quad I_0(x) = 1 + \sum_{k=1}^{\infty} \left[\frac{1}{k!}\left(\frac{x}{2}\right)^k\right]^2$$

and $M = (N-1)/2$. Its frequency spectrum is given by

$$W_K(e^{j\omega T}) = w_K(0) + 2\sum_{n=1}^{M} w_K(nT)\cos\omega nT$$

An FIR LP filter that would satisfy the specifications shown can be readily designed by using a procedure due to Kaiser as detailed in the next three slides.

1. Determine the impulse response $h(nT)$ using the Fourier series assuming an idealized frequency response

$$H(e^{j\omega T}) = \begin{cases} 1 & \text{for } |\omega| \leq \omega_c \\ 0 & \text{for } \omega_c < |\omega| \leq \omega_s/2 \end{cases} \quad \text{where } \omega_c = \frac{1}{2}(\omega_p + \omega_a)$$

(see Eq. (B).)

1. Determine the impulse response $h(nT)$ using the Fourier series assuming an idealized frequency response

$$H(e^{j\omega T}) = \begin{cases} 1 & \text{for } |\omega| \leq \omega_c \\ 0 & \text{for } \omega_c < |\omega| \leq \omega_s/2 \end{cases} \quad \text{where } \omega_c = \frac{1}{2}(\omega_p + \omega_a)$$

(see Eq. (B).)

2. Choose δ such that the actual passband ripple, A_p, is equal to or less than specified passband ripple, $\tilde{A}_p$, and the actual minimum stopband attenuation, A_a, is equal or greater than the specified minimum stopband attenuation, $\tilde{A}_a$.

A suitable value is

$$\delta = \min\,(\tilde{\delta}_p, \tilde{\delta}_a)$$

$$\text{where} \quad \tilde{\delta}_p = \frac{10^{0.05\tilde{A}_p} - 1}{10^{0.05\tilde{A}_p} + 1} \quad \text{and} \quad \tilde{\delta}_a = 10^{-0.05\tilde{A}_a}$$

3. With the required δ defined, the actual stopband attenuation A_a can be calculated as

$$A_a = -20 \log_{10} \delta$$

3. With the required δ defined, the actual stopband attenuation A_a can be calculated as

$$A_a = -20 \log_{10} \delta$$

4. Choose parameter α as

$$\alpha = \begin{cases} 0 & \text{for } A_a \leq 21 \\ 0.5842(A_a - 21)^{0.4} + 0.07886(A_a - 21) & \text{for } 21 < A_a \leq 50 \\ 0.1102(A_a - 8.7) & \text{for } A_a > 50 \end{cases}$$

5. Choose parameter D as

$$D = \begin{cases} 0.9222 & \text{for } A_a \leq 21 \\ \dfrac{A_a - 7.95}{14.36} & \text{for } A_a > 21 \end{cases}$$

Then select the lowest odd value of N that would satisfy the inequality

$$N \geq \frac{\omega_s D}{B_t} + 1 \quad \text{where} \quad B_t = \omega_a - \omega_p$$

5. Choose parameter D as

$$D = \begin{cases} 0.9222 & \text{for } A_a \leq 21 \\ \dfrac{A_a - 7.95}{14.36} & \text{for } A_a > 21 \end{cases}$$

Then select the lowest odd value of N that would satisfy the inequality

$$N \geq \frac{\omega_s D}{B_t} + 1 \quad \text{where } B_t = \omega_a - \omega_p$$

6. Form $w_K(nT)$ using Eq. (D).

5. Choose parameter D as

$$D = \begin{cases} 0.9222 & \text{for } A_a \leq 21 \\ \dfrac{A_a - 7.95}{14.36} & \text{for } A_a > 21 \end{cases}$$

Then select the lowest odd value of N that would satisfy the inequality

$$N \geq \frac{\omega_s D}{B_t} + 1 \quad \text{where } B_t = \omega_a - \omega_p$$

6. Form $w_K(nT)$ using Eq. (D).

7. Form

$$H'_w(z) = z^{-M} H_w(z) \quad \text{where } H_w(z) = \mathcal{Z}[w_K(nT)h(nT)]$$

and $M = (N-1)/2$.

- The design method presented can be easily extended to FIR HP, BP, and BS filters.

- Consider the case where a BP filter is required that would satisfy the following specifications:

  - Passband ripple $\leq \tilde{A}_p$
  - Minimum stopband attenuation $\geq \tilde{A}_a$
  - Lower passband edge ω_{p1}
  - Upper passband edge ω_{p2}
  - Lower stopband edge ω_{a1}
  - Upper stopband edge ω_{a2}
  - Sampling frequency ω_{s2}

  ($\tilde{A}_p$ and $\tilde{A}_a$ in dB and all frequencies in rad/s)

A. Antoniou
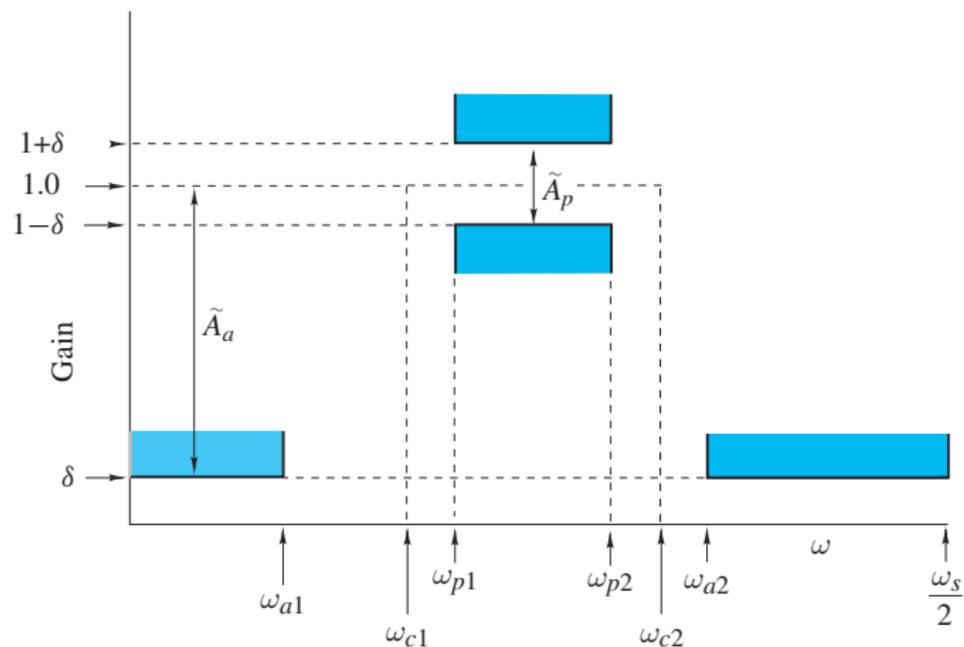
The only differences in the design of BP filters are to

- use the more critical of the two transition widths for the design,

- use the idealized frequency of a BP filter for the determination of the initial impulse response.

The more critical transition width is

$$B_t = \min[(\omega_{p1} - \omega_{a1}),\ (\omega_{a2} - \omega_{p2})]$$

and hence the idealized frequency response for a BP filter is deduced as

$$H(e^{j\omega T}) = \begin{cases} 1 & \text{for } -\omega_{c2} \le \omega \le -\omega_{c1} \\ 1 & \text{for } \omega_{c1} \le \omega \le \omega_{c2} \\ 0 & \text{otherwise} \end{cases}$$

where
$$\omega_{c1} = \omega_{p1} - \frac{B_t}{2} \quad \omega_{c2} = \omega_{p2} + \frac{B_t}{2}$$

Example

Design an FIR BP filter satisfying the following specifications:

- Minimum attenuation for $0 \leq \omega \leq 200$: 45 dB

- Maximum passband ripple for $400 < \omega < 600$: 0.2 dB

- Minimum attenuation for $700 \leq \omega \leq 1000$: 45 dB

- Sampling frequency: 2000 rad/s

The idealized impulse response is obtained as

$$h(nT) = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} H(e^{j\omega T}) e^{j\omega nT} \, d\omega$$

$$= \frac{1}{n\pi} (\sin \omega_{c2} nT - \sin \omega_{c1} nT)$$

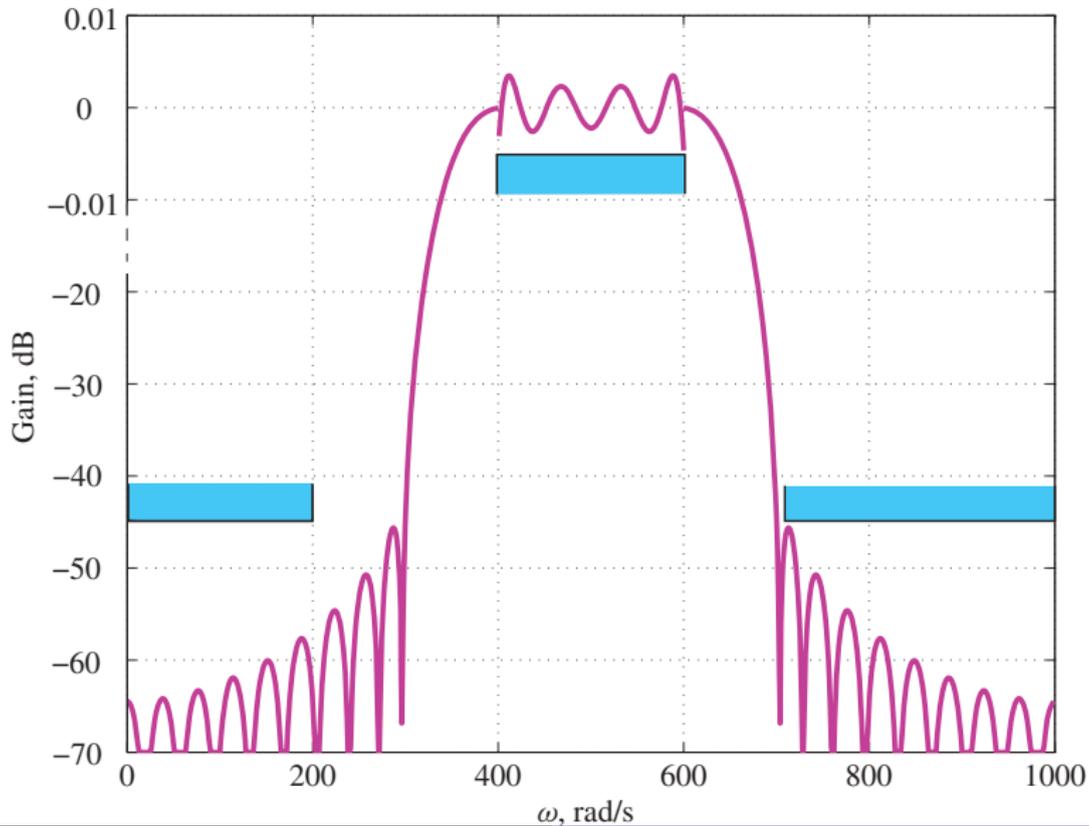The application of the design procedure described will give

$$\alpha = 3.9754$$
$$D = 2.580$$
$$N = 53$$

- A more recent approach for the design of FIR filters that parallels the method described is a method based on the ultraspherical window function proposed by Bergen and Antoniou (see References).

- A more recent approach for the design of FIR filters that parallels the method described is a method based on the ultraspherical window function proposed by Bergen and Antoniou (see References).

- For certain specifications, this new approach tends to give more efficient designs, i.e., the minimum filter length that will achieve the required specifications is somewhat lower.

A DSP software package that incorporates the design techniques described in this presentation is *D-Filter*. Please see

http://www.d-filter.ece.uvic.ca

for more information.

- A method for the design of FIR filters using a procedure proposed by Kaiser has been described.

- A method for the design of FIR filters using a procedure proposed by Kaiser has been described.

- The method is easy to apply and requires a minimal amount of computation.

  Hence it can be used to design filters on-line in real or quasi-real time applications.

  The method is implemented in a DSP software package known as D-Filter.

- A method for the design of FIR filters using a procedure proposed by Kaiser has been described.

- The method is easy to apply and requires a minimal amount of computation.

  Hence it can be used to design filters on-line in real or quasi-real time applications.

  The method is implemented in a DSP software package known as D-Filter.

- The designs obtained are suboptimal, i.e., other methods are available that would yield a lower filter order for the same specifications, for example, the weighted-Chebyshev method which will be described in Part 2 of this tutorial.

References

- A. Antoniou, *Digital Signal Processing: Signals, Systems, and Filters*, Chap. 9, McGraw-Hill, 2005.

- J. F. Kaiser, "Nonrecursive digital filter design using the I_0-sinh window function," *in Proc. IEEE Int. Symp. Circuit Theory, 1974*, pp. 20–23.

- S. W. A. Bergen and A. Antoniou, "Design of ultraspherical window functions with prescribed spectral characteristics," *EURASIP Journal on Applied Signal Processing*, vol. 13, pp. 2053-2065, 2004.

- S. W. A. Bergen and A. Antoniou, "Design of nonrecursive digital filters using the ultraspherical window function," *EURASIP Journal on Applied Signal Processing*, vol. 13, pp. 1910-1922, 2005.

- T. Saramäki, "Adjustable windows for the design of FIR filters — A tutorial," 6th Mediterranean Electrotechnical Conference, vol. 1, pp. 28–33, May 1991.

- R. L. Streit, "A two-parameter family of weights for nonrecursive digital filters and antennas," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 32, pp. 108–118, Feb. 1984.

- A. Antoniou, "Design of digital differentiators satisfying prescribed specifications," *Proc. Inst. Elect. Eng.*, Part E, vol. 127, pp. 24–30, Jan. 1980.

This slide concludes the presentation.

Thank you for your attention.